Abstract

Let P be a set of points in $\mathbb{R}^d$ and let $\mathcal{F}$ be the family of all distinct objects of a particular kind (hyperspheres, boxes, simplices, ...), such that each object in $\mathcal{F}$ has a distinct tuple of points from P on its boundary. For ex., in $d = 2$, $\mathcal{F}$ could be the family of $\binom{n}{2}$ axis-parallel rectangles such that each rectangle has a distinct pair of points of P forming its diagonal. $\mathcal{F}$ is called the set of all objects induced (spanned) by P. We consider various hitting and piercing problems for some families of geometrical objects induced by a point set.

- **Selection Lemma** - What is the largest subset of $\mathcal{F}$ that can be pierced by a single point ?
- **Minimum Hitting set** - What is the minimum set of points in P needed to hit all the objects in $\mathcal{F}$?

Selection Lemma type results typically bound the maximum number of induced objects that are pierced by a single point. Selection Lemmas are classical results in discrete geometry that have been well studied and have applications in many geometric problems like weak epsilon nets and slimming Delaunay triangulations.

For the *First Selection Lemma*, we consider the set of all distinct induced objects of a particular kind. The first selection lemma for triangles in $\mathbb{R}^2$ showed that there exists a point that is present in $\frac{2}{9} \cdot \binom{n}{3}$ (constant fraction of) triangles induced by P. Moreover, the constant in this result is tight. This question has also been considered extensively for induced simplices in $\mathbb{R}^d$.

We explore this question for other geometric objects like axis-parallel boxes and hyperspheres in $\mathbb{R}^d$ in the thesis. We prove a tight bound of $\frac{n^2}{8}$ for axis-parallel rectangles

induced from two points. We also explore this question for induced axis-parallel boxes and hyperspheres in higher dimensions and obtain non-trivial bounds.

We also look at a generalization of the first selection lemma, known as *Second Selection Lemma*, which considers an m-sized arbitrary subset $\mathcal{S} \subseteq \mathcal{F}$ of distinct induced objects of a particular kind and shows that there exists a point which is contained in $f(m,n)$ objects of $\mathcal{S}$. Firstly, we prove almost tight bounds for induced intervals in $\mathbb{R}$. We then obtain bounds of $\frac{m^3}{24n^4}$ for the second selection lemma of axis-parallel rectangles. This improves upon the previous bounds of $\Omega\left(\frac{m^2}{n^2 \log^2 n}\right)$ by Smorodinsky and Sharir (2004), when there are near-quadratic number of induced rectangles.

Finally, we consider the second category of questions i.e. the *Hitting Set Problem* for induced objects. This is a special case of the geometric hitting set problem which has been extensively studied and is known to be NP-Hard for objects like lines and unit disks and even APX-Hard for axis-parallel slabs and rectangles in $\mathbb{R}^2$, unit balls in $\mathbb{R}^3$ etc. In particular, we show that the minimum hitting set problem for the set of all induced lines is NP-Complete by a reduction from the Multi-Color Clique problem. We also look at an abstract set system generalization of the induced lines and prove that the hitting set problem for this set system is NP-Complete by a similar reduction.