

Abstract

In this thesis we study the following dimensional parameters: boxicity, cubicity, threshold dimension and poset dimension. While the first three parameters are defined on graphs, poset dimension is defined on partially ordered sets (or posets). We only consider finite graphs and posets. In addition, we assume that the graphs are simple and undirected.

Boxicity and Cubicity: A k -box (k -cube) is a Cartesian product of closed intervals (unit-intervals) $[a_1, b_1] \times \cdots \times [a_k, b_k]$. The boxicity (cubicity) of a graph G , $\text{box}(G)$ ($\text{cub}(G)$) is the minimum integer k such that every vertex in G is mapped to a k -box (k -cube) in the k -dimensional Euclidean space and two boxes (cubes) intersect if and only if their corresponding vertices are adjacent in G . Boxicity and cubicity can be considered as extensions of the concept of interval graphs and unit-interval graphs respectively.

Threshold Dimension: A graph G is a *threshold graph* if there is a real number p and a weight function $w : V \rightarrow \mathbb{R}$ such that for any two vertices $u, v \in V(G)$, $\{u, v\}$ is an edge if and only if $w(u) + w(v) \geq p$. The threshold dimension of a graph G is the minimum integer k such that there exist k threshold graphs G_i , $i = 1, 2, \dots, k$ which satisfy $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k)$.

Poset Dimension: Let $\mathcal{P} = (S, P)$ be a poset where S is a finite non-empty set and P is a reflexive, anti-symmetric and transitive binary relation on S . P is a *total order* if every pair of elements in S is comparable in P . The *dimension* of \mathcal{P} , denoted by $\text{dim}(\mathcal{P})$ is the minimum integer k such that there exist k total orders on S , L_1, \dots, L_k and for two distinct elements $x, y \in S$: $x < y$ in P if and only if $x < y$ in each L_i , $i \in \{1, 2, \dots, k\}$.

All the four dimensional parameters that we have considered are very hard to compute. It is NP-complete to even determine if the boxicity of a graph is at most 2, if its cubicity is at most 3, if its threshold dimension is at most 3 and if the dimension of a poset is at most 3. Also it is hard to design an approximation algorithm within \sqrt{n} factor for computing the dimension of a poset.

Our Results We state some of our main results:

1. Lower bounds for boxicity: We have developed two general methods based on certain vertex isoperimetric properties of graphs for deriving lower bounds. Application of these methods has led to some significant results. We mention a few of them here: (a) Almost all graphs have boxicity $\Omega(n)$. (b) For a fixed k , boxicity of random k -regular graphs is $\Omega(k/\log k)$.
2. Consider a poset $\mathcal{P} = (S, P)$ and let $G_{\mathcal{P}}$ be its underlying comparability graph. We show that for any poset \mathcal{P} , $\text{box}(G_{\mathcal{P}})/(\chi(G_{\mathcal{P}}) - 1) \leq \dim(\mathcal{P}) \leq 2\text{box}(G_{\mathcal{P}})$, where $\chi(G_{\mathcal{P}})$ is the chromatic number of $G_{\mathcal{P}}$ and $\chi(G_{\mathcal{P}}) \neq 1$. Some important consequences of this result are: (a) It allows us to derive hitherto unknown upper bounds for poset dimension such as $\dim(\mathcal{P}) \leq 2\text{treewidth}(G_{\mathcal{P}}) + 4$. (b) The boxicity of any graph with maximum degree Δ is $O(\Delta \log^2 \Delta)$ which is an improvement over the best known upper bound of $\Delta^2 + 2$. (c) There exist graphs with boxicity $\Omega(\Delta \log \Delta)$. This disproves a conjecture that the boxicity of a graph is $O(\Delta)$. (d) There exists no polynomial-time algorithm to approximate the boxicity of a bipartite graph on n vertices within a factor of $O(n^{0.5-\epsilon})$ for any $\epsilon > 0$, unless $NP = ZPP$.
3. We show that every poset can be associated with a split graph such that the threshold dimension of the complement of the split graph is equal to the dimension of the poset. As a consequence we show that there exists no polynomial-time algorithm to approximate the threshold dimension of a split graph on n vertices with a factor of $O(n^{0.5-\epsilon})$ for any $\epsilon > 0$, unless $NP = ZPP$.
4. We have given an upper bound for the cubicity of interval graphs. *Claw number* of a graph G , $\psi(G)$ is the largest positive integer m such that $K_{1,m}$ is an induced subgraph of G . If G is an interval graph, we show that $\lceil \log_2 \psi(G) \rceil \leq \text{cub}(G) \leq \min(\lceil \log_2 \alpha \rceil, \lceil \log_2 \psi(G) \rceil + 2)$, where α is the independence number of G .
5. We have improved upper bounds for the dimension of *incidence posets* and *interval orders* which are among the well-studied classes of posets.